MAT1332 additional practice problems

The practice problems here are in addition to the material given in the suggested exercises, the assignments and the tests.
The problems here do NOT represent a sample exam.

1. Find the volume of the solid obtained by rotating the area bounded by $y = 4x - x^2$, $y = 3$, $x = 1$ and $x = 3$ about the x-axis.

   Sol. See the following Figure 1,

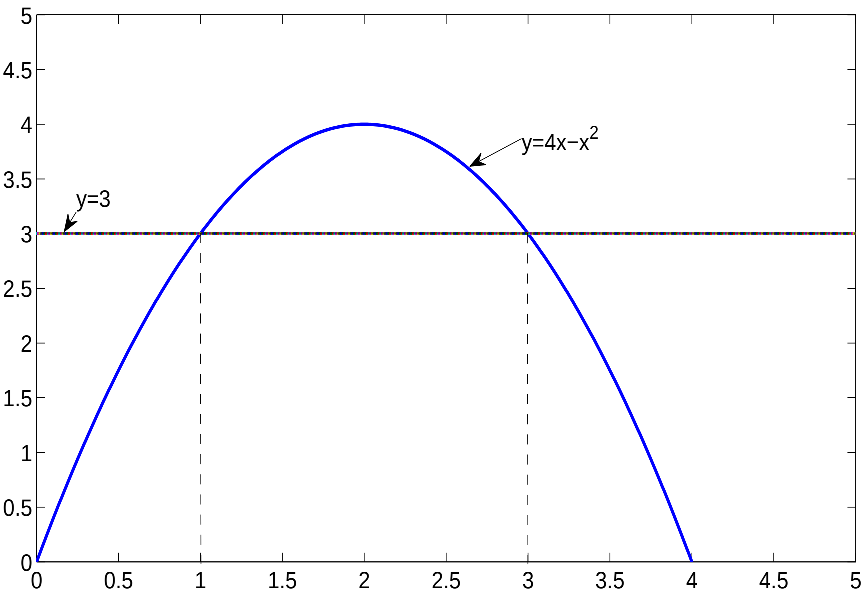

Figure 1: The illustrative graph of Question 1.

denote $A_1(x) = 4x - x^2$ and $A_2(x) = 3$, we know that The volume of the solid is given by

$$
\begin{aligned}
V &= V_1 - V_2 \\
&= \int_1^3 \pi (A_1(x))^2 dx - \int_1^3 \pi (A_2(x))^2 dx \\
&= \int_1^3 \pi (4x - x^2)^2 dx - \int_1^3 \pi \cdot 3^2 dx \\
&= \int_1^3 \pi (16x^2 - 8x^3 + x^4) dx - \int_1^3 9\pi dx \\
&= \pi (\frac{16}{3}x^3 - 2x^4 + \frac{1}{5}x^5)\Big|_1^3 - 9\pi x\Big|_1^3 \\
&= \frac{136}{15}\pi.
\end{aligned}
$$

2. Consider the system

$$
\begin{aligned}
x' &= \sqrt{3}x + 2y \\
y' &= 11x - \sqrt{3}y
\end{aligned}
$$

(a) Show that $(0,0)$ is the only equilibrium.
(b) Write down the Jacobian matrix.
(c) Show that the eigenvalues are $\lambda = \pm 5$.
(d) For each eigenvalue, find the corresponding eigenvectors.

Sol.

(a) Set $x' = 0$ and $y' = 0$, we have

$$
\begin{aligned}
\sqrt{3}x + 2y &= 0 \\
11x - \sqrt{3}y &= 0
\end{aligned}
$$

Solving the above linear system for x and y, we obtain the unique solution $x = 0$ and $y = 0$.

(b) The Jacobian matrix is given by

$$J = \begin{bmatrix} \sqrt{3} & 2 \\ 11 & -\sqrt{3} \end{bmatrix}.$$

(c) Note that $\mathrm{tr}(J) = 0$, and $\det(J) = -25$, so the eigenvalues are given by

$$\lambda = \frac{\mathrm{tr}(J)}{2} \pm \sqrt{\frac{(\mathrm{tr}(J))^2}{4} - \det(J)} = \pm\sqrt{25} = \pm 5.$$

(d) For $\lambda_1 = 5$, we have

$$\left[\begin{array}{cc|c} \sqrt{3}-5 & 2 & 0 \\ 11 & -\sqrt{3}-5 & 0 \end{array}\right] \xrightarrow{(\sqrt{3}+5)R_1} \left[\begin{array}{cc|c} -22 & 2(\sqrt{3}+5) & 0 \\ 11 & -\sqrt{3}-5 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1}$$

$$\left[\begin{array}{cc|c} -11 & \sqrt{3}+5 & 0 \\ 11 & -\sqrt{3}-5 & 0 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{cc|c} -11 & \sqrt{3}+5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Set x_2 as the free variable, i.e., $x_2 = t$, from the first row, we have $-11x_1 + (\sqrt{3}+5)x_2 = 0$ to solve for x_1, then $x_1 = \frac{\sqrt{3}+5}{11}t$. Thus, the corresponding eigenvector is

$$v_1 = \begin{bmatrix} \frac{\sqrt{3}+5}{11} \\ 1 \end{bmatrix} t, \quad (t \neq 0).$$

For $\lambda_2 = -5$, we have

$$\left[\begin{array}{cc|c} \sqrt{3}+5 & 2 & 0 \\ 11 & -\sqrt{3}+5 & 0 \end{array}\right] \xrightarrow{(\sqrt{3}-5)R_1} \left[\begin{array}{cc|c} -22 & 2(\sqrt{3}-5) & 0 \\ 11 & -\sqrt{3}+5 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1}$$

$$\left[\begin{array}{cc|c} -11 & \sqrt{3}-5 & 0 \\ 11 & -\sqrt{3}+5 & 0 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{cc|c} -11 & \sqrt{3}-5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Set x_2 as the free variable, i.e., $x_2 = t$, from the first row, we have $-11x_1 + (\sqrt{3}-5)x_2 = 0$ to solve for x_1, then $x_1 = \frac{\sqrt{3}-5}{11}t$. Thus, the corresponding eigenvector is

$$v_2 = \begin{bmatrix} \frac{\sqrt{3}-5}{11} \\ 1 \end{bmatrix} t, \quad (t \neq 0).$$

3. Evaluate $\int_1^4 e^{\sqrt{x}}dx$,

Sol. Let $u = \sqrt{x}$, then $du = \frac{1}{2}\frac{1}{\sqrt{x}}dx$, i.e. $dx = 2udu$. The integration limits are changed from $x = 1$ and $x = 4$ to $u = 1$ and $u = 2$. So by integration by parts, we have

$$\begin{aligned}\int_1^4 e^{\sqrt{x}}dx &= \int_1^2 e^u \cdot 2udu = 2\int_1^2 ude^u = 2ue^u\Big|_1^2 - 2\int_1^2 e^u du \\ &= 2ue^u\Big|_1^2 - 2e^u\Big|_1^2 = 2e^2.\end{aligned}$$

4. Evaluate $\int_1^2 \ln(x^2e^x)dx$,

Sol.

$$\begin{aligned}\int_1^2 \ln(x^2e^x)dx &= \int_1^2 (\ln x^2 + \ln e^x)dx = \int_1^2 (2\ln x + x)dx = 2\int_1^2 \ln x dx + \int_1^2 x dx \\ &= 2x\ln x\Big|_1^2 - 2\int_1^2 x \cdot \frac{1}{x}dx + \int_1^2 xdx = 2x\ln x\Big|_1^2 - 2\int_1^2 dx + \int_1^2 xdx \\ &= 2x\ln x\Big|_1^2 - 2x\Big|_1^2 + \frac{1}{2}x^2\Big|_1^2 = 4\ln 2 - \frac{1}{2}.\end{aligned}$$

5. Evaluate $\int \frac{2x-1}{(x+4)(x+1)}dx$,

Sol.

$$\frac{2x-1}{(x+4)(x+1)} = \frac{A}{x+4} + \frac{B}{x+1} = \frac{A(x+1)+B(x+4)}{(x+4)(x+1)} = \frac{(A+B)x+(A+4B)}{(x+4)(x+1)}$$

Setting $A + B = 2$ and $A + 4B = -1$, we have $A = 3$ and $B = -1$. Hence

$$\int \frac{2x-1}{(x+4)(x+1)}dx = \int \frac{3}{x+4} + \frac{-1}{x+1}dx = 3\ln|x+4| - \ln|x+1| + C.$$

6. Evaluate $\int \frac{x^2+1}{x^2+3x+2}dx$,

Sol. Note that $x^2 + 3x + 2 = (x+1)(x+2)$, and $x^2 + 1 = (x^2 + 3x + 2) - 3x - 1$, thus

$$\int \frac{x^2+1}{x^2+3x+2}dx = \int \frac{(x^2+3x+2)-3x-1}{x^2+3x+2} = \int dx - \int \frac{3x+1}{x^2+3x+2}dx$$

However,

$$\frac{3x+1}{x^2+3x+2} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{A(x+2)+B(x+1)}{(x+1)(x+2)} = \frac{(A+B)x+(2A+B)}{(x+1)(x+2)}$$

Setting $A + B = 3$ and $2A + B = 1$, we have $A = -2$, and $B = 5$. Finally

$$\begin{aligned}\int \frac{x^2+1}{x^2+3x+2}dx &= \int dx - \int \frac{3x+1}{x^2+3x+2}dx \\ &= \int dx - \int \frac{-2}{x+1} + \frac{5}{x+2}dx \\ &= x + 2\ln|x+1| - 5\ln|x+2| + C.\end{aligned}$$

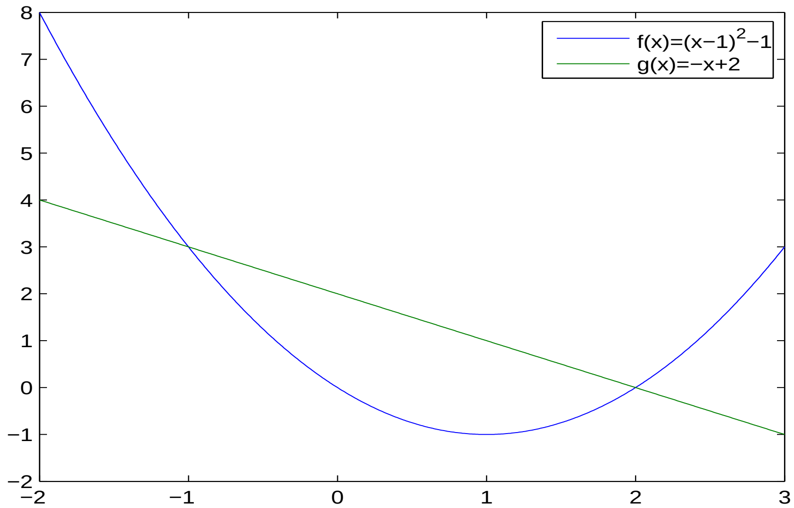

Figure 2: The graphs of $f(x) = (x-1)^2 - 1$ and $g(x) = -x + 2$ in Question 7.

7. Find the area enclosed by the following two functions $f(x) = (x-1)^2 - 1,\ g(x) = -x + 2$.

   Sol. Find the bounding curves are graphed in Figure 2.

   To find the points where the two curves intersect, we solve

$$(x-1)^2 - 1 = -x + 2 \Longleftrightarrow (x+1)(x-2) = 0$$

   Therefore, $x = -1$ and $x = 2$ are the x-coordinates of the points of intersection. The area of enclosed by $f(x)$ and $g(x)$ is given by

$$\begin{aligned}
\int_{-1}^{2} |f(x) - g(x)| dx &= \int_{-1}^{2} [g(x) - f(x)] dx \\
&= \int_{-1}^{2} [-x + 2 - (x-1)^2 + 1] dx \\
&= \int_{-1}^{2} [-x + 2 - x^2 + 2x - 1 + 1] dx \\
&= \int_{-1}^{2} [-x^2 + x + 2] dx = [-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x]\Big|_{-1}^{2} = \frac{9}{2}.
\end{aligned}$$

8. Does the integral $\int_0^2 \frac{1}{(1-x)^{1/3}} dx$ converge? If so, to which value?

   Sol.

$$\begin{aligned}
\int_0^2 \frac{1}{(1-x)^{1/3}} dx &= \int_0^1 \frac{1}{(1-x)^{1/3}} dx + \int_1^2 \frac{1}{(1-x)^{1/3}} dx \\
&= -\frac{3}{2}(1-x)^{\frac{2}{3}}\Big|_0^1 - \frac{3}{2}(1-x)^{\frac{2}{3}}\Big|_1^2 \\
&= -0 + \frac{3}{2} - \frac{3}{2} + 0 = 0.
\end{aligned}$$

9. Does the integral $\int_0^2 \frac{1}{(x-1)^4} dx$ converge? If so, to which value?

Sol.

$$\begin{aligned}\int_0^2 \frac{1}{(x-1)^4}dx &= \int_0^1 \frac{1}{(x-1)^4}dx + \int_1^2 \frac{1}{(x-1)^4}dx \\ &= -\frac{1}{3}(x-1)^{-3}\Big|_0^1 - \frac{1}{3}(x-1)^{-3}\Big|_1^2 \\ &= -\infty - \frac{1}{3} - \frac{1}{3} + \infty\end{aligned}$$

So this integral doesn't converge.

10. Does the integral $\int_0^2 \frac{1}{(x-1)^{2/5}}dx$ converge? If so, to which value?

Sol.

$$\begin{aligned}\int_0^2 \frac{1}{(x-1)^{2/5}}dx &= \int_0^1 \frac{1}{(x-1)^{2/5}}dx + \int_1^2 \frac{1}{(x-1)^{2/5}}dx \\ &= \frac{5}{3}(x-1)^{\frac{3}{5}}\Big|_0^1 + \frac{5}{3}(x-1)^{\frac{3}{5}}\Big|_1^2 \\ &= 0 - \frac{5}{3} + \frac{5}{3} - 0 = 0.\end{aligned}$$

11. Does the integral $\int_0^\infty \frac{1}{\sqrt{x+1}}dx$ converge? If so, to which value?

Sol.

$$\int_0^\infty \frac{1}{\sqrt{x+1}}dx = 2(x+1)^{\frac{1}{2}}\Big|_0^\infty = \infty - 2$$

So this integral is divergent.

12. Solve $\frac{dx}{dt} = \sqrt{3t+1}$ with $x(0) = 1$.

Sol. This differential equation is a pure time differential equation.

$$\begin{aligned}\frac{dx}{dt} &= \sqrt{3t+1} \\ dx &= \sqrt{3t+1}dt \\ \int dx &= \int \sqrt{3t+1}dt \\ x &= \frac{1}{3}\cdot\frac{2}{3}(3t+1)^{\frac{3}{2}} + C \\ x(t) &= \frac{2}{9}(3t+1)^{\frac{3}{2}} + C\end{aligned}$$

Applying I.C. $x(0) = 1$, we obtain $C = \frac{7}{9}$. Thus the solution is

$$x(t) = \frac{2}{9}(3t+1)^{\frac{3}{2}} + \frac{7}{9}.$$

13. Solve $\frac{dx}{dt} = 5 - 16t^2$ with $x(3) = -11$.

Sol. This differential equation is a pure time differential equation.

$$\begin{aligned}
\frac{dx}{dt} &= 5 - 16t^2 \\
dx &= (5 - 16t^2)dt \\
\int dx &= \int (5 - 16t^2)dt \\
x &= 5t - \frac{16}{3}t^3 + C
\end{aligned}$$

Applying I. C. $x(3) = -11$, we obtain $C = 118$. Thus the solution is

$$x(t) = 5t - \frac{16}{3}t^3 + 118.$$

14. Solve $\frac{dy}{dx} = 3x^2 e^{2y}$ with $y(0) = 0$.

    Sol. This non-autonomous differential equation is separable.

$$\begin{aligned}
\frac{dy}{dx} &= 3x^2 e^{2y} \\
e^{-2y}dy &= 3x^2 dx \\
\int e^{-2y}dy &= \int 3x^2 dx \\
-\frac{1}{2}e^{-2y} &= x^3 + C \\
e^{-2y} &= -2x^3 - 2C \\
-2y &= \ln(-2x^3 - 2C) \\
y &= -\frac{1}{2}\ln(-2x^3 - 2C)
\end{aligned}$$

Applying I. C. $y(0) = 0$, we obtain $C = -\frac{1}{2}$. So the solution is

$$y(x) = -\frac{1}{2}\ln(1 - 2x^3).$$

15. Solve $\frac{dy}{dx} = \frac{2x}{y+e^{5y}}$ with $y(2) = 0$.

    Sol. This non-autonomous differential equation is separable.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{2x}{y + e^{5y}} \\
(y + e^{5y})dy &= 2x dx \\
\int (y + e^{5y})dy &= \int 2x dx \\
\frac{y^2}{2} + \frac{e^{5y}}{5} &= x^2 + C
\end{aligned}$$

Applying I. C. $y(2) = 0$, we get $C = -\frac{19}{5}$. So the solution is implicitly given by

$$\frac{y^2}{2} + \frac{e^{5y}}{5} = x^2 - \frac{19}{5}.$$

16. Suppose that

$$\frac{dy}{dx} = y(1-y)(y-2)$$

(a) Find the equilibria of this differential equation.

(b) Graph $\frac{dy}{dx}$ as a function of y, and use your graph to discuss the stability of the equilibria.

(c) Draw the phase-line diagram.

(d) Use the derivative test to discuss the stability of equilibria.

(e) Sketch the solution with the initial conditions $y(0) = 0.5$ and $Y(0) = 1.5$ respectively.

Sol. This differential equation is autonomous.

(a) The equilibria of this differential equation are given by solving

$$y(1-y)(y-2) = 0 \Longleftrightarrow y_1 = 0,\ y_2 = 1,\ y_3 = 2.$$

(b) See function $\frac{dy}{dx}$ as a function of y is graphed in the following Figure 3. From the graph of $\frac{dy}{dx}$, we

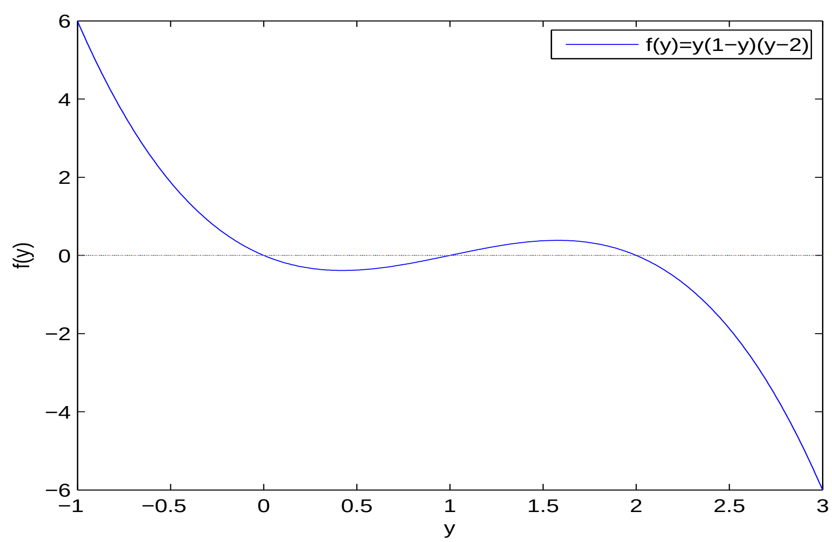

Figure 3: The graph of $\frac{dy}{dx}$ as a function of y in part (b) of Question 16.

know that, the slope at $y_1 = 0$ is negative, so $y_1 = 0$ should be stable; the slope at $y_2 = 1$ is positive, so $y_2 = 1$ should be unstable; the slope at $y_3 = 2$ is negative, so $y_3 = 2$ should be stable.

(c) The phase-line diagram is given in the following Figure 4.

(d) Let $f(y) = y(1-y)(y-2)$, i.e., $f(y) = -y^3 + 3y^2 - 2y$. Then $f'(y) = -3y^2 + 6y - 2$ and

$$\begin{aligned} f'(y_1) &= f'(0) = -2 < 0, \text{ then } y_1 = 0 \text{ is stable;} \\ f'(y_2) &= f'(1) = 1 > 0, \text{ then } y_2 = 1 \text{ is unstable;} \\ f'(y_3) &= f'(2) = -2 < 0, \text{ then } y_3 = 2 \text{ is stable.} \end{aligned}$$

(e) The solution with initial conditions $y(0) = 0.5$ and $y(0) = 1.5$ are sketched in the following Figure 5 respectively.

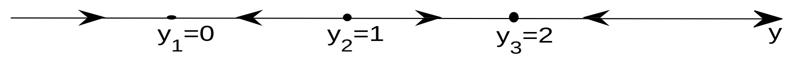

Figure 4: The phase-line diagram in part (c) of Question 16.

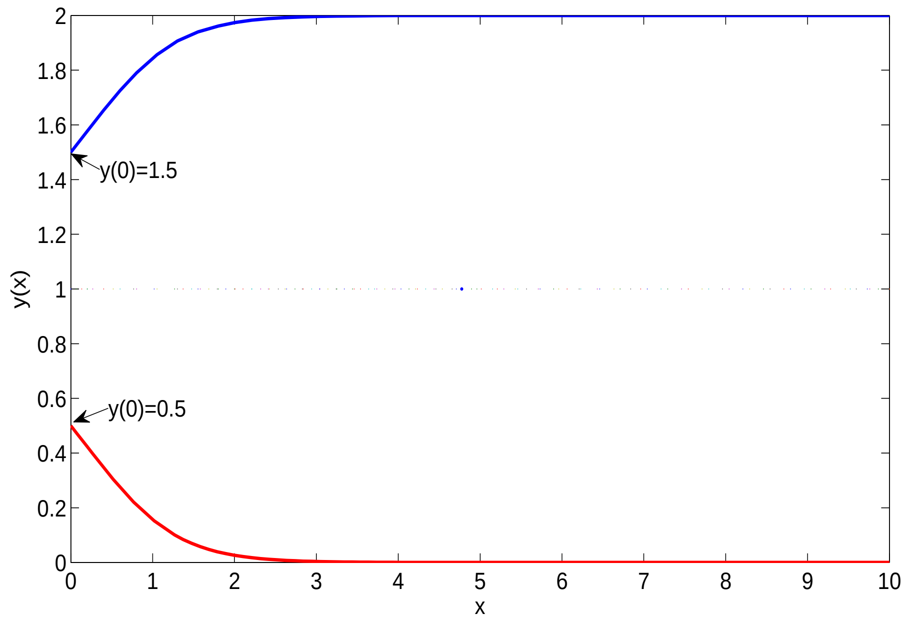

Figure 5: The solution curves with initial conditions $y(0) = 0.5$ and $y(0) = 1.5$ in part (e) of Question 16.

17. Considering the following predator-prey system:

$$\begin{cases} \frac{dN}{dt} = aN(1 - \frac{N}{K}) - bNP \\ \frac{dP}{dt} = cNP - dP \end{cases}$$

where $N = N(t)$ is the prey density at time t and $P = P(t)$ is the predator density at time t. The constants a, b, c, d and K are positive. Furthermore, assume that $d/c < K$.

(a) Find all of equilibrium points.

(b) Find the Jacobian Matrix of the system in general.

(c) Evaluate the Jacobian Matrix at the nontrivial equilibrium (N^*, P^*) with $N^* > 0$ and $P^* > 0$, find its eigenvalues and determine whether this point is stable or not.

Sol.

(a) The equilibrium points will be given by solving

$$\begin{aligned} aN(1-\frac{N}{K})-bNP &= 0 \Longleftrightarrow N[a(1-\frac{N}{K})-bP]=0 \Longleftrightarrow N=0 \text{ or } a(1-\frac{N}{K})-bP=0 \\ cNP-dP &= 0 \Longleftrightarrow (cN-d)p=0 \Longleftrightarrow N=\frac{d}{c} \text{ or } P=0 \end{aligned}$$

So all of equilibrium points are

$$\begin{aligned} (N_1,P_1) &= (0,0) \\ (N_2,P_2) &= \Big(\frac{d}{c},\frac{a}{b}(1-\frac{1}{K}\frac{d}{c})\Big), \\ (N_3,P_3) &= (K,0). \end{aligned}$$

(b) Let $f(N,P)=aN(1-\frac{N}{K})-bNP$, and then $f(N,P)=aN-\frac{a}{K}N^2-bNP$, and $g(N,P)=cNP-dP$. Thus the Jacobian Matrix of the system in general is given by

$$J(N,P)=\begin{bmatrix} a-2\frac{a}{K}N-bP & -bN \\ cP & cN-d \end{bmatrix}.$$

(c) Since $d/c<K$, then $P_2=\frac{a}{b}(1-\frac{1}{K}\frac{d}{c})>0$, thus (N_2,P_2) is the only nontrivial equilibrium. Then evaluating the Jacobian Matrix $J(N,P)$ at (N_2,P_2), we have

$$\begin{aligned} J(N_2,P_2) &= \begin{bmatrix} a-2\frac{a}{K}N_2-bP_2 & -bN_2 \\ cP_2 & cN_2-d \end{bmatrix} = \begin{bmatrix} a-2\frac{a}{K}\frac{d}{c}-b\frac{a}{b}(1-\frac{1}{K}\frac{d}{c}) & -b\frac{d}{c} \\ c\frac{a}{b}(1-\frac{1}{K}\frac{d}{c}) & c\frac{d}{c}-d \end{bmatrix} \\ &= \begin{bmatrix} -\frac{ad}{Kc} & -\frac{bd}{c} \\ \frac{ac}{b}-\frac{ad}{bK} & 0 \end{bmatrix}. \end{aligned}$$

Note that $\mathrm{tr}(J(N_2,P_2))=-\frac{ad}{Kc}$ and $\det(J(N_2,P_2))=\frac{bd}{c}(\frac{ac}{b}-\frac{ad}{bK})=ad-\frac{ad^2}{cK}$, then the eigenvalues are given by

$$\begin{aligned} \lambda &= \frac{\mathrm{tr}(J(N_2,P_2))}{2}\pm\sqrt{\frac{(\mathrm{tr}(J(N_2,P_2)))^2}{4}-\det(J(N_2,P_2))} \\ &= -\frac{1}{2}\frac{ad}{Kc}\pm\sqrt{\frac{1}{4}\frac{(ad)^2}{(Kc)^2}-ad+\frac{ad^2}{cK}} \\ &= -\frac{1}{2}\frac{ad}{Kc}\pm\sqrt{\frac{1}{4}\frac{(ad)^2}{(Kc)^2}-ad(1-\frac{d}{c}\frac{1}{K})}. \end{aligned}$$

Note that $d/c<K$ gives us that $1-\frac{d}{c}\frac{1}{K}>0$, and then $\frac{1}{4}\frac{(ad)^2}{(Kc)^2}-ad(1-\frac{d}{c}\frac{1}{K})<\frac{1}{4}\frac{(ad)^2}{(Kc)^2}$, which implies that λ will always have negative real parts. Hence, this nontrivial equilibrium (N_2,P_2) is stable.

18. Find the tangent plane to the surface

$$z=f(x,y)=4x^2+y^2$$

at the point $(1,2,8)$.

Sol. The partial derivatives are given by

$$\begin{aligned}\frac{\partial f}{\partial x} &= 8x\\ \frac{\partial f}{\partial y} &= 2y\end{aligned}$$

$z_0 = f(1,2) = 4\cdot 1^2 + 2^2 = 8$, and $\frac{\partial f}{\partial x}(1,2) = 8$, $\frac{\partial f}{\partial y}(1,2) = 4$, thus the tangent plane is given by

$$\begin{aligned} z - z_0 &= \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)\\ z-8 &= 8(x-1)+4(y-2)\\ z &= 8x+4y-8.\end{aligned}$$

19. Let

$$A = \begin{bmatrix} -4 & -2\\ 1 & -1\end{bmatrix}$$

(a) Show that the eigenvalues of A are -3 and -2.
(b) Find the solution of the system of differential equations

$$\begin{aligned}\frac{dx}{dt} &= -4x(t) - 2y(t)\\ \frac{dy}{dt} &= x(t) - y(t)\end{aligned}$$

if $x(0) = 1$ and $y(0) = 2$.
(c) Draw the phase portrait of the system of differential equations given in (b).

Sol.

(a) Note that $\mathrm{tr}(A) = -4 + (-1) = -5$ and $\det(A) = (-4)\cdot(-1) - (-2)\cdot 1 = 6$. So the eigenvalues of A are given by

$$\lambda = \frac{\mathrm{tr}(A)}{2} \pm \sqrt{\frac{(\mathrm{tr}(A))^2}{4} - \det(A)} = \frac{-5}{2} \pm \sqrt{\frac{(-5)^2}{4} - 6} = \frac{-5}{2} \pm \frac{1}{2}.$$

Hence, $\lambda_1 = -3$ and $\lambda_2 = -2$.

(b) For $\lambda_1 = -3$, we have

$$\left[\begin{array}{cc|c} -4+3 & -2 & 0\\ 1 & -1+3 & 0\end{array}\right] \to \left[\begin{array}{cc|c} -1 & -2 & 0\\ 1 & 2 & 0\end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{cc|c} -1 & -2 & 0\\ 0 & 0 & 0\end{array}\right]$$

Set x_2 free, then $x_2 = t$, from the first row, we have, $-x_1 - 2x_2 = 0$. Then $x_1 = -2t$. So the corresponding eigenvector is given by

$$v_1 = \begin{bmatrix} -2\\ 1\end{bmatrix}.$$

For $\lambda_1 = -2$, we have

$$\left[\begin{array}{cc|c} -4+2 & -2 & 0\\ 1 & -1+2 & 0\end{array}\right] \to \left[\begin{array}{cc|c} -2 & -2 & 0\\ 1 & 1 & 0\end{array}\right] \xrightarrow{-\frac{1}{2}*R_1+R_2} \left[\begin{array}{cc|c} -2 & -2 & 0\\ 0 & 0 & 0\end{array}\right]$$

Set x_2 free, then $x_2 = t$, from the first row, we have, $-2x_1 - 2x_2 = 0$. Then $x_1 = -t$. So the corresponding eigenvector is given by

$$v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The general solution is given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{-3t} v_1 + C_2 e^{-2t} v_2 = C_1 e^{-3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Applying I.C., we have

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2C_1 - C_2 \\ C_1 + C_2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

Solving this gives us that $C_1 = -3$ and $C_2 = 5$. The solution is finally given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{-3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -3e^{-3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + 5e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6e^{-3t} - 5e^{-2t} \\ -3e^{-3t} + 5e^{-2t} \end{bmatrix}.$$

(c) The phase portrait of the system of differential equations given in (b) is graphed in the following Figure 6.

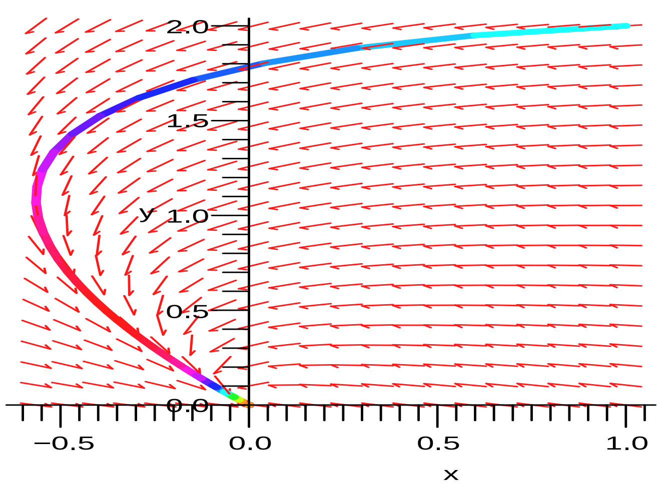

Figure 6: The phase portrait in part (c) of Question 19.

20. Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & -5 \\ 2 & 5 & -6 \end{bmatrix}$. Solve the system of linear equations $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Sol. Note that $\det(A) = 1 \cdot 3 \cdot (-6) + 1 \cdot (-5) \cdot 2 - (-1) \cdot 3 \cdot 2 - 1 \cdot (-5) \cdot 5 = 3 \neq 0$, so Matrix A is invertible.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 3 & -5 & 0 & 1 & 0 \\ 2 & 5 & -6 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(-2)*R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 3 & -5 & 0 & 1 & 0 \\ 0 & 3 & -4 & -2 & 0 & 1 \end{array}\right] \xrightarrow{(-1)*R_2+R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 3 & -5 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right] \xrightarrow{R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & -1 & 1 \\ 0 & 3 & -5 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right] \xrightarrow{5*R_3+R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & -1 & 1 \\ 0 & 3 & 0 & -10 & -4 & 5 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right] \xrightarrow{\frac{1}{3}*R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & -\frac{10}{3} & -\frac{4}{3} & \frac{5}{3} \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right] \xrightarrow{(-1)*R_2+R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{3} & \frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 0 & -\frac{10}{3} & -\frac{4}{3} & \frac{5}{3} \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right]$$

Thus

$$A^{-1} = \begin{bmatrix} \frac{7}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{10}{3} & -\frac{4}{3} & \frac{5}{3} \\ -2 & -1 & 1 \end{bmatrix}.$$

The solution of linear equations is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{10}{3} & -\frac{4}{3} & \frac{5}{3} \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ -\frac{5}{3} \\ -1 \end{bmatrix}.$$